Richardson-Lucy Algorithm for image restoration

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This document explains Richardson-Lucy algorithm [Lucy, 1974; Richardson, 1972] which is used for image restoration.

1 Imaging equation

Assuming Poisson noise, imaging equation is derived as

\[ g(x) = \text{Poisson}(f(x) \otimes h(x)), \]  

(1)

where \( g, f, \) and \( h \) are a blurred image, the latent image, and the Point Spread Function (PSF) respectively. \text{Poisson} operator adds Poisson noise to the image.

2 Bayes’ theorem

Given a blurred image \( g \), a posterior distribution is

\[ P(f(x)|g(x)) = \frac{P(g(x)|f(x))P(f(x))}{P(g(x))}, \]  

(2)

where \( P(g(x)|f(x)), P(g(x)), \) and \( P(f(x)) \) are the likelihood, the evidence, and the prior distribution.

3 Poisson noise likelihood

When image noise follows Poisson distribution, the likelihood is formulated as

\[ P(g(x)|f(x)) = \prod_x \frac{f(x) \otimes h(x)^g(x) \exp (-f(x) \otimes h(x))}{g(x)!}. \]  

(3)

Maximization of the likelihood is equivalent to minimization of its negative logarithm. Therefore, the negative
log likelihood $L(f(x))$ is

$$L(f(x)) = -\ln(P(g(x)|f(x))) = \sum_x \ln \frac{f(x) \otimes h(x)^{g(x)} \exp(-f(x) \otimes h(x))}{g(x)!} = \sum_x f(x) \otimes h(x) - g(x) \ln(f(x) \otimes h(x)) + \ln g(x)!.$$  

(4)

(5)

(6)

Consider a small perturbation $\Delta x$. The negative log likelihood $L(f(x + \Delta x))$ is

$$L(f(x + \Delta x)) = \sum_x f(x + \Delta x) \otimes h(x) - g(x) \ln((f(x + \Delta x) \otimes h(x)) + g(x)!.$$ (7)

Removing constant term w.r.t. $f$, we obtain

$$L(f(x + \Delta x)) = \sum_x f(x + \Delta x) \otimes h(x) - g(x) \ln(f(x + \Delta x) \otimes h(x))$$

(8)

Assuming $f(x + \Delta x) = f(x) + f(\Delta x)$,

$$L(f(x + \Delta x)) = \sum_x (f(x) + f(\Delta x)) \otimes h(x) - g(x) \ln((f(x) + f(\Delta x)) \otimes h(x))$$

$$= \sum_x f \otimes h(x) + f \otimes h(\Delta x) - g(x) \ln(f \otimes h(x) + f \otimes h(\Delta x)).$$ (9)

For simplicity,

$$f \otimes h(x) = f(x) \otimes h(x),$$

$$f \otimes h(\Delta x) = f(\Delta x) \otimes h(x).$$

$$L(f(x + \Delta x)) = \sum_x f \otimes h(x) + f \otimes h(\Delta x) - g(x) \ln(f \otimes h(x) + f \otimes h(\Delta x))$$

$$= \sum_x f \otimes h(x) + f \otimes h(\Delta x) - g(x) \ln \left[ f \otimes h(x) \left( 1 + \frac{f \otimes h(\Delta x)}{f \otimes h(x)} \right) \right]$$

$$= \sum_x f \otimes h(x) + f \otimes h(\Delta x) - g(x) \ln \left( 1 + \frac{f \otimes h(\Delta x)}{f \otimes h(x)} \right)$$

$$= L(f(x)) + \sum_x f \otimes h(\Delta x) - g(x) \ln \left( 1 + \frac{f \otimes h(\Delta x)}{f \otimes h(x)} \right).$$

Following Taylor expansion $\ln(1 + x) \approx x - \frac{x^2}{2}$,

$$L(f(x + \Delta x)) = L(f(x)) + \sum_x f \otimes h(\Delta x) - g(x) \ln \left( 1 + \frac{f \otimes h(\Delta x)}{f \otimes h(x)} \right)$$

$$= L(f(x)) + \sum_x f \otimes h(\Delta x) - g(x) \frac{f \otimes h(\Delta x)}{f \otimes h(x)} + \frac{1}{2} g(x) \left( \frac{f \otimes h(\Delta x)}{f \otimes h(x)} \right)^2.$$
Omitting the last term, because it’s too small,

\[ L(f(x + \Delta x)) = L(f(x)) + \sum_x f \otimes h(\Delta x) - g(x) \frac{\int f \otimes h(x) dx}{\int f \otimes h(x)} \]

\[ = L(f(x)) + \sum_x f \otimes h(\Delta x) \left( 1 - \frac{g(x)}{f \otimes h(x)} \right). \]

From the definition of convolution integral \( \int ab \otimes cdx = \int ba \otimes hdx \), where \( h^* \) is the adjoint of \( h \),

\[ L(f(x + \Delta x)) = L(f(x)) + \sum_x f \otimes h(\Delta x) \left( 1 - \frac{g(x)}{f \otimes h(x)} \right) \]

\[ = L(f(x)) + \sum_x \left( 1 - \frac{g(x)}{f \otimes h(x)} \right) \otimes \bar{h}(\Delta x). \]

The partial derivative of \( L(f(x)) \) on \( x \) is derived as

\[ \frac{\partial L(f(x))}{\partial x} = \frac{L(f(x + \Delta x)) - L(f(x))}{\Delta x} \]

\[ = \frac{1}{\Delta x} \sum_x \left( 1 - \frac{g(x)}{f \otimes h(x)} \right) \otimes \bar{h}. \]

Since the minimization of the negative log likelihood is obtained by finding \( x \) satisfying

\[ \frac{\partial L(f(x))}{\partial x} = 0. \]

Thus, we have

\[ \left( 1 - \frac{g(x)}{f \otimes h(x)} \right) \otimes \bar{h} = 0 \]

\[ 1 - \frac{g(x)}{f \otimes h(x)} \otimes \bar{h} = 0 \]

(10)

Using the convergence condition \( \frac{f^{n+1}}{f^n} = 1 \), we obtain the update rule as

\[ \frac{f^{n+1}}{f^n} = \frac{g(x)}{f \otimes h(x)} \otimes \bar{h}. \]

(11)

Finally, we obtain the Richardson-Lucy deconvolution algorithm as

\[ f^{n+1} = \left( \frac{g(x)}{f \otimes h(x)} \otimes \bar{h} \right) f^n. \]

(12)

**Bibliography**
